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ANALYTIC REPRESENTATION OF MEMBER FORCES IN LINEAR ELASTIC REDUNDANT TRUSSES

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Abstract—The process of designing a truss of given geometry and material properties, where the design variables are the cross-sectional areas of the bars is hampered by the need to reanalyse the structure many times until an acceptable design is obtained. Currently, approximate explicit analysis models, based on truncated linear Taylor series expansions, are used to evaluate the structural response at the various candidate design points. Due to the approximate nature of the analysis model, the structure is designed iteratively until convergence of both the analysis equations and the design process.

This paper presents for the first time the exact analytic expressions of the internal loads in a truss which is subjected to static loads. The stress resultants are the ratio of two multilinear polynomials in the element stiffness. The number of terms of the polynomials is equal to the number of combinations of statically determinate stable structures which can be derived from the original structures. The coefficients of the polynomial expansions can be obtained from equilibrium considerations and from enforcing "global" compatibility of deformations. The expressions are explicit in both the external loads and the element stiffness.

The applicability of the analytic equations hinges on the number of combinations of statically determinate stable substructures. In the case of small size structures, the present explicit equations circumvent the need for approximate reanalysis. In common engineering structures, the number of stable subsets is prohibitively large, which renders the analytic expressions intractable. The exact analytic expressions may, however, constitute a starting point for constructing approximate explicit analysis equations of improved quality.

1. INTRODUCTION

Consider a linear elastic truss of given geometry consisting of M members and N nodal degrees of freedom ($M \geqslant N$), which is subjected to an N-vector of static loads p applied at the nodes of the structure. Let s and t be respectively the M-vectors of element axial stiffnesses ($s_j = E_jA_j/L_j$) and element internal loads, where E_j, A_j and L_j are respectively Young's modulus, the cross-sectional area and the length of element j. The governing equations for analyzing the structure are:

(1) Statics: $$Qt = p \tag{1a}$$

(2) Constitutive law: $$Se = t \tag{1b}$$

(3) Kinematics: $$Ru = e, \tag{1c}$$

where Q is the $(N \times M)$ Statics matrix, S is the $(M \times M)$ diagonal stiffness matrix ($S_{ij} = \delta_{ij}s_j$, where δ_{ij} is the Kronecker delta), e is the M-vector of element total elongations, R is the $(M \times N)$ Kinematics matrix and u is the N-vector of nodal displacements. Clebsch's theorem (Asplund, 1966) establishes the duality between the statics and kinematics equations through $Q^T = R$.

Equations (1) are traditionally solved by either the Flexibility method or the Stiffness method. In the sequel we will concentrate on the Flexibility method in which the element loads are the principal unknowns. If the structure is statically determinate ($M = N$), the element loads can be obtained directly from the statics equations, provided that the Statics matrix is non-singular. If the structure is statically redundant ($M > N$) the N statics equations are augmented with $R(=M-N)$ compatibility equations which are obtained from the kinematics equations [see, for instance, Fuchs (1981)]

$$e_1 = R_1 R_0^{-1} e_0, \tag{2}$$

where subscripts 0 and 1 refer to a subset of statically determinate and redundant elements respectively. Substituting eqns (1a) and (1b) in (2) and eliminating the loads t_0 of the basic structure, yields the so-called equations of consistent deformations. These are R equations in the redundant forces t_1 and are usually obtained from structural considerations by performing cuts at one end of every redundant bar and by requiring compatibility of displacements on both sides of the cut section. The equations of consistent deformations can be written as follows

$$Ct_1 + d = 0, \tag{3}$$

with

$$C_{ij} = \sum_{\substack{k=1 \\ k \neq i}}^{N} \frac{n_{ik} n_{jk}}{s_k} + \delta_{ij} \frac{1}{s_i} \quad i, j = 1, \ldots, R, \tag{4}$$

and

$$d_i = \sum_{k=1}^{N} \frac{n_{ik} t_k}{s_k} \quad i = 1, \ldots, R, \tag{5}$$

where the summations are carried out for the bars of the statically determinate substructure. In these equations, n_{mk} is the load in bar k due to a pair of unit and opposite forces applied at both sides of the cut section in bar m, and t_k is the load in bar k due to the applied external forces. Note, the entries of the compatibility matrix C and of the relative displacement vector d depend on s the stiffnesses of the structure. The above equations yield the forces in the redundant bars. The forces in the determinate basic structure are obtained from the equilibrium eqn (1a)

$$Q_0 t_0 = p - Q_1 t_1. \tag{6}$$

Consider the task of designing a statically redundant truss of given topology and nodal coordinates. That is, we seek to determine the stiffness of the elements (usually only the cross-sections are variables) in order for the structure to carry safely and economically a set of applied loads. In the absence of a theory (and equations) of design the only available approach is by trial and error. Based on former experience and engineering judgement, an initial set of stiffnesses is assumed and an analysis of the structure is performed. This involves the computation of the elements of matrix C and vector d followed by the solution of eqns (3) and (6). Usually the results will show that the structure is either overstressed and too flexible (not safe) or understressed and too stiff (not economical). In a subsequent iteration the stiffnesses will be modified in order to obtain improved results, and a reanalysis of the structure is performed. This procedure is repeated until a satisfactory design is obtained. Every reanalysis requires the re-computation of C and d and the solution of eqns (3).

The question of multiple reanalysis has been a subject of concern for generations of structural engineers and many ingeneous techniques have been proposed over the years. The problem has become even more acute since computerized methods for structural design have emerged [see for instance Kirsch (1981)] in which the structural design problem is solved via mathematical programming methods. These techniques are in fact trial and error methods embedded in an efficient algorithmic logic. Traditionally if one would perform only a few structural reanalyses, with mathematical programming methods the number of required structural reanalyses have increased by orders of many magnitude.

From the onset of computerized design it was clear that without explicit analysis equations, that is, an explicit expression of the forces in the bars (or the nodal displacements) as a function of the design variables (the cross-sections) the mathematical programming approach would run out of steam. The El Dorado of the structural designer became in some sense the explicit inverse of the compatibility matrix C or of the stiffness matrix if the displacement method was employed. The exact explicit analysis equations were never found and research was channeled towards devizing approximate explicit analysis expressions. Reviews of work in the field of approximate reanalysis were published by Arora (1976) and more recently by Abu Kassim and Topping (1985). Since the results of the analysis model are approximate, the structure is now optimized iteratively, where at the end of each minimization cycle the approximate analysis model is updated and a new minimization process is performed.

This paper presents for the first time the exact analytic expression of the forces in the bars of a redundant truss as a function of the stiffnesses of the elements. The force in a bar of a truss is the ratio of two multilinear polynomials of order N in the stiffnesses of the structure. The number of terms in each polynomial is equal to the number of statically determinate stable substructures which can be derived from the structure. The coefficients of the polynomials in the numerator can be obtained through equilibrium considerations. The coefficients of the polynomial in the denominator, which is common to all forces, can be computed by enforcing compatibility.

As mentioned, the number of terms in the N-polynomials is equal to the number of statically determinate stable substructures which can be obtained from the original truss. Unfortunately, for common engineering structures, the set of stable substructures is prohibitively large. The analytic expressions in their present formulation are therefore applicable only to relatively small sized structures. For the automated design of more complex structures one must still resort to some form of approximation.

2. THE GENERAL ANALYTIC EXPRESSION OF THE INTERNAL FORCES

Let us assume that the degree of statical redundancy of the structure is R and let us solve the compatibility eqns (3) by Cramer's rule (Ayers, 1962). The force in a redundant bar j is the ratio of the determinants of matrices C_j and C

$$t_j = \frac{|C_j|}{|C|}, \tag{7}$$

where C_j is the matrix obtained from C by replacing its jth column with vector $-d$. The purpose of this section is to establish that both determinants in eqn (7) are multilinear polynomials of degree N in the stiffnesses of the structure. A typical term in the expansion of the determinants will be shown to be equal to a constant multiplying the product of N stiffnesses. Having no inclination to prefer one set of N stiffnesses over a different set, all possible combinations of N stiffnesses out of the pool of M stiffnesses will be included in the expansion. Subsequent sections will describe a method to compute the constants of the polynomials.

We will start with the determinant of C. By definition, a determinant is equal to the sum of all the combinations of signed products, each term of which containing one and only one element from any row and one and only one element of any column of the matrix. Matrix C being of the order R, a typical term in the expansion of the determinant is therefore

$$\pm C_{ij} C_{kl} C_{mn}, \ldots \text{ (R elements)}, \tag{8}$$

where C_{ij}, C_{kl} and C_{mn} are entries in matrix C such that row indices (column indices) are never repeated in any given term. For example, C_{ij} and C_{im} cannot both appear in one term since we would have two elements of row i in this term, which is in violation of the definition of the determinant.

Note, every element of matrix C (4) is itself a polynomial in $1/s_j$, since the member forces due to unit loads, n_{ik} and n_{jk}, do not depend on the stiffnesses. If we use subscripts $a, b, c, \ldots$ for bars in the basic structure and subscripts $i, j, k, \ldots$ for redundant bars, eqn (8) can be further detailed as

$$\pm\left(\frac{n_{ia}n_{ja}}{s_a}+\frac{n_{ib}n_{jb}}{s_b}+\frac{n_{ic}n_{jc}}{s_c}+\ldots\right)\left(\frac{n_{ka}n_{la}}{s_a}+\frac{n_{kb}n_{lb}}{s_b}+\frac{n_{kc}n_{lc}}{s_c}+\ldots\right)$$

$$\times\left(\frac{1}{s_m}+\frac{n_{ma}n_{ma}}{s_a}+\frac{n_{mb}n_{mb}}{s_b}+\frac{n_{mc}n_{mc}}{s_c}+\ldots\right)\ldots(R \text{ elements}). \quad (9)$$

Please bear in mind that what we have here is one typical term of the expansion of the determinant. Carrying out the multiplications in expression (9) we obtain terms of the form

$$\frac{n_{ia}n_{ja}n_{ka}n_{la}1\ldots}{s_a s_a s_m\ldots} \quad (10a)$$

$$\frac{n_{ia}n_{ja}n_{kb}n_{lb}1\ldots}{s_a s_b s_m\ldots} \quad (10b)$$

and so on. Note, we always have a constant in the numerators and a product of R stiffnesses in denominators. When we now muster all the terms of the type given in eqns (10) resulting from the expansion of determinant $|C|$ we obtain an almost hopeless series of the form

$$|C| = \sum_k B_k/\mu_k, \quad (11)$$

where the B_ks are constants and the μ_ks are products of the form

$$\mu_k = s_a^p s_b^q s_c^r \ldots s_1^u s_2^v \ldots, \quad (12)$$

where the exponents $(p, q, r\ldots)$ of the basic bars are positive or zero integers and the exponents $(u, v\ldots)$ of the redundant elements are equal to zero or one. The products μ_k are of the order R in the stiffnesses.

The determinant of C_j has a similar expansion

$$|C_j| = \sum_k A_{jk}/\mu_k, \quad (13)$$

where the A_{jk}s are coefficients which are independent of the stiffnesses.

The complexity of these results can be drastically reduced by noting that μ_k must be linear in every stiffness which appears in the product. In other words, the exponents of the stiffnesses in eqn (12) are either one or zero. Consequently every product μ_k will contain R distinct stiffnesses. Terms of the form (10a) for instance, where s_a in the denominator is non-linear, will not be part of the expansion of the determinant. To show this we will use the standard technique of the method of compatible displacements, considering one redundant member force only. Let us disconnect an arbitrary bar i of the original redundant structure, and let us apply at both ends of the cut section in equal and opposite fashion the released redundant force t_i. The compatibility equation for the relative displacements at the cut section is

$$c_{ii}t_i + d_i = 0, \quad (14)$$

with

$$c_{ii} = \sum_{\substack{k=1\\k\neq i}}^{M} \frac{n_{ik}^2}{s_k} + \frac{1}{s_i} \tag{15}$$

$$d_i = \sum_{k=1}^{M} \frac{n_{ik}t_k}{s_k}, \tag{16}$$

where n_{ik} is the force in bar k due to a pair of unit and opposite loads applied at the cut section and t_k is the force in bar k due to a pair of unit and opposite loads applied at the cut section and t_k is the force in bar k due to the external loads. Note, the "released" structure may still be statically indeterminate, and therefore the n_{ik}s and the t_ks are usually a function of the stiffnesses of the structure. However, they do not depend on the stiffness s_i of the cut element. By keeping the stiffnesses of all the bars except bar i constant, we obtain the following expression for the load in bar i

$$t_i = \frac{c_i}{d_i + 1/s_i}, \tag{17}$$

where c_i and d_i are independent of s_i. Now, the force in an arbitrary bar j is the superposition of t_j due to the external forces and the load due to a pair of equal and opposite forces t_i applied at both ends of the cut section

$$t_{ji} = t_j + n_{ij}t_i \tag{18}$$

which in conjunction with eqn (17) yields

$$t_{ji} = \frac{(c_i n_{ij} + d_i t_j) + t_j/s_i}{d_i + 1/s_i} \tag{19}$$

with t_{ji} denoting the expression for the force in member j when member i alone is allowed to vary. All the coefficients of this equation are independent of s_i. The numerator and the denominator of eqn (19) will be recognized as the expansions of the determinants in eqn (7) when all the stiffnesses but s_i are kept constant. Since this result holds for any bar i, the stiffnesses in the expression of μ_k (11) can only be equal to one or zero (when they are excluded from the particular μ_k). Consequently every term in the expression of μ_k is now the linear product of R stiffnesses.

Introducing eqns (11) and (13) in eqn (7) and multiplying the numerator and denominator by the product of the stiffnesses of all the M bars of the structure we obtain the general expression for the load in an arbitrary bar j:

$$t_j = \frac{\sum_k A_{jk}\pi_k}{\sum_k B_k \pi_k}, \quad k = 1, \ldots, C_N^M, \tag{20}$$

where the number of terms in each series is equal to the number of different combinations C_N^M of a subset of N bars which can be selected out of a total of M bars

$$C_N^M = \frac{M!}{N!R!}, \tag{21}$$

and every π_k is the product of the stiffnesses of the N bars in that subset

$$\pi_k = s_i s_j \ldots s_m \quad (N \text{ terms}). \tag{22}$$

The constants A_{jk} and B_k in eqn (20) can be determined from considerations of equilibrium and compatibility as will be demonstrated in the following sections.

3. USING EQUILIBRIUM TO DETERMINE THE A_{jk} CONSTANTS

To determine the A_{jk} constants, let us extract from the general eqn (20) the expression for the load t_{jm} in bar j for the statically determinate subset m. To do so, we set to zero all the bars which do not appear in the subset m. Since all the π_ks except π_m reduce to zero, eqn (20) gives

$$t_{jm} = \frac{A_{jm}\pi_m}{B_j\pi_m}, \tag{23}$$

and therefore

$$A_{jm} = t_{jm}B_j. \tag{24}$$

As expected the stiffnesses cancel out from eqn (23) since the forces in a statically determinate structure are independent of the rigidities. An additional result which can be drawn from eqn (23) is that combinations of N bars which are not stable must not appear in the summations of eqn (20). If they appear, eqn (23) would be able to predict the forces in a non-stable structure, which is inconceivable. We have thus reduced the general expression of the force in a bar of a truss to the structured expression

$$t_j = \frac{\sum_k B_k t_{jk} \pi_k}{\sum_k B_k \pi_k}, \tag{25}$$

where the summations are now carried out over all the statically determinate stable substructures which can be obtained from the redundant truss.

One could argue that unstable structures (with N bars) which can carry a particular loading condition should be present in eqn (25) for the case of that loading condition. However, kinematically unstable combinations are not structures, in the pure sense of the word, and are therefore not part of eqn (25). Consequently, eqn (25) will be singular for a conditionally stable structure.

To visualize the results obtained so far let us apply the theory to the three-bar truss of Fig. 1. This structure has been studied extensively by investigators in the field of structural design. It is composed of three bars with two nodal degrees of freedom, and it therefore has a statical redundancy of $R = 1$. Young's modulus for all the bars is E. The design variables of the structure are the stiffnesses of the three bars, s_1, s_2 and s_3. By inspection we find that the truss has three statically determinate stable substructures composed of

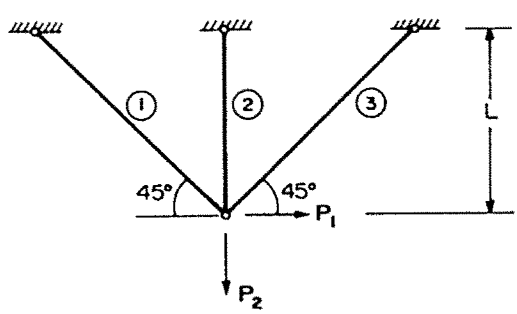

Fig. 1. The three-bar truss problem.

Table 1

Substructure	Internal loads in bars $[t_{jk}]$			B_k
Missing bar $[k]$	1	2	3	
1	*	(p_1+p_2)	$-\sqrt{2}p_1$	1
2	$\sqrt{2}/2(p_1+p_2)$	*	$\sqrt{2}/2(p_1-p_1)$	2
3	$\sqrt{2}p_1$	(p_2-p_1)	*	1

members (2, 3), (1, 3) and (1, 2) respectively. In the following, index k will denote the substructure from which bar k is missing. Using eqn (25) we can immediately write the expression of the member loads of the three-bar truss

$$t_1 = \frac{1}{\Delta}(B_1t_{11}s_2s_3+B_2t_{12}s_1s_3+B_3t_{13}s_1s_2)$$

$$t_2 = \frac{1}{\Delta}(B_1t_{21}s_2s_3+B_2t_{22}s_1s_3+B_3t_{23}s_1s_2)$$

$$t_3 = \frac{1}{\Delta}(B_1t_{31}s_2s_3+B_2t_{32}s_1s_3+B_3t_{33}s_1s_2) \quad (26)$$

with

$$\Delta = B_1s_2s_3+B_2s_1s_3+B_3s_1s_2 \quad (27)$$

where the member forces t_{jk} (force in member j if the loads are applied to substructure k only) are given in Table 1 in the case of a general loading p_1, p_2. What remains to be computed are the three B_k constants. As will be shown in the next section, these constants can be obtained from compatibility conditions.

4. USING COMPATIBILITY TO COMPUTE THE B_k CONSTANTS

Up to this point we have used the statics equations (1a) and the material constitutive law (1b) to derive the general expression (25) of the internal forces in a truss. The equilibrium equations are present in the t_{jk}s and the element properties (stiffnesses) appear in the π_ks. We will now show that the B_k coefficients can be obtained from the kinematics equations (1c). The method which will be presented may not be the only way to determine these coefficients and for that matter, it may not be the most efficient approach. As will be shown, the B_k coefficients represent compatibility of deformations. Without loss of generality, the method will be introduced for the three-bar truss.

Eliminating the displacement vector **u** from the kinematics equations (1c) yields the compatibility equation which expresses the deformation of a redundant bar in terms of the deformations of the determinate bars:

$$e_1-\sqrt{2}e_2+e_3 = 0. \quad (28)$$

By dividing both sides of expression t_j ($j = 1, 2, 3$) in eqns (26) by s_j, and using the values of t_{jk} given in Table 1, we obtain the expression of the elongations of the bars of the structure

$$e_1 = \frac{1}{\Delta}\left[\frac{\sqrt{2}}{2}(p_1+p_2)B_2s_3+\sqrt{2}p_1B_3s_2\right]$$

$$e_2 = \frac{1}{\Delta}[(p_1+p_2)B_1s_3+(p_2-p_1)B_3s_1]$$

$$e_3 = \frac{1}{\Delta}\left[-\sqrt{2}p_1B_1s_2+\frac{\sqrt{2}}{2}(p_2-p_1)B_2s_1\right] \quad (29)$$

which when introduced in the compatibility equation (28), and after grouping the terms in s_1, s_2 and s_3 yields the condition

$$\left(\frac{B_2}{2}-B_3\right)s_1+(B_3-B_1)s_2+\left(\frac{B_2}{2}-B_1\right)s_3=0. \tag{30}$$

Since this is a "universal" compatibility equation, that is, it enforces compatibility for any values of the stiffnesses, the terms in parenthesis must be identically zero. From the structure of eqns (25) we note that one of the B_ks can be given an arbitrary value. Setting $B_1 = 1$ for instance, condition (30) yields an overdetermined but consistent set of linear equations in the B_ks with the result shown in the last column of Table 1.

It is worthwhile to emphasize that the external loading p does not appear in the conditions for compatibility given by eqn (30). This is no coincidence. As is shown in the Appendix, the compatibility coefficients B_k depend only on the geometry of the structure, which is to be expected since they derive from purely kinematic considerations. In other words, the three conditions embedded in eqn (30) are the same for any external loading.

In the more general case, the procedure to determine the B_k coefficients is numerically more complex but the concept remains unchanged. Consider a redundant structure consisting of M bars with N nodal degrees of freedom. In every π_k in eqn (25) we will have the products of subsets of N stiffnesses out of the available M bars. The expression of the deformations of the elements is obtained by dividing every product π_k appearing in the numerator of a force in eqn (25) by its corresponding stiffness

$$e_i=\frac{\sum_k B_k t_{ik}\pi_k/s_i}{\sum_k B_k\pi_k} \tag{31}$$

These elongations are then introduced in compatibility equations given by eqn (2). Since the compatibility equations are linear and homogeneous expressions in e, the denominator in eqns (31) will cancel out. What we are left with are homogeneous polynomials in terms of combinations of products of $(N-1)$ stiffnesses (combinations of π_k/s_i). The coefficients multiplying these products of stiffnesses are linear homogeneous polynomials in terms of the B_k constants. In order to satisfy the compatibility equations for any value of the stiffnesses, all the B_k polynomials must be identically zero. This leads to an overdetermined set of homogeneous linear equations in the B_ks. This overdetermined set of equations is consistent since no simplifying assumptions or approximations were made during the derivation of the present theory. By giving an arbitrary value to one of the constants (say $B_1 = 1$) the remaining unknowns can be determined. As mentioned earlier, the values of the B_ks are independent of the external loading.

In the next section the method to determine the analytic expression of the internal loads in a redundant truss will be illustrated by a numerical example.

5. A NUMERICAL EXAMPLE

The ten-bar truss depicted in Fig. 2 is a classic test case of optimal structural design (Kirsch, 1981). The truss is composed of $M = 10$ bars and has $N = 8$ nodal degrees of freedom, the degree of static redundancy is thus $R = 2$. Young's modulus is the same for all the bars of the structure. The structure is subjected to two vertical loads of intensity p, applied at the nodes of the bottom chord. The number of different combinations of 8 bars out of a total of 10 bars is 45 (eqn 21), but only 29 of them lead to stable statically determinate substructures. The stable substructures are shown in Fig. 3 where every combination is denoted by a sequential number on the top-left of the structure, and by the pair of missing redundant bars on the top-right of the structure (note, index 0 stands for bar

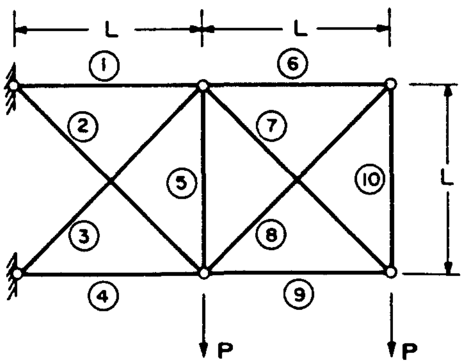

Fig. 2. The ten-bar truss problem.

10). For example, B_{25} corresponds to the substructures from which bars 2 and 5 are missing. Table 2 gives the values of the internal forces in the ten elements for the external loads applied directly to the 29 substructures. At this point, the loads in bars 1 and 2, for instance, are

$$t_1 = \frac{3B_{25}\pi_{25} + 3B_{26}\pi_{26} + 3B_{27}\pi_{27} + 3B_{28}\pi_{28} + \cdots + 2B_{50}\pi_{50}}{B_{15}\pi_{15} + B_{16}\pi_{16} + B_{17}\pi_{17} + B_{18}\pi_{18} + \cdots + B_{50}\pi_{50}}$$

$$t_2 = \frac{3\sqrt{2}B_{15}\pi_{15} + 3\sqrt{2}B_{16}\pi_{16} + 3\sqrt{2}B_{17}\pi_{17} + 3\sqrt{2}B_{18}\pi_{18} + \cdots + \sqrt{2}B_{50}\pi_{50}}{B_{15}\pi_{15} + B_{16}\pi_{16} + B_{17}\pi_{17} + B_{18}\pi_{18} + \cdots + B_{50}\pi_{50}}. \quad (32)$$

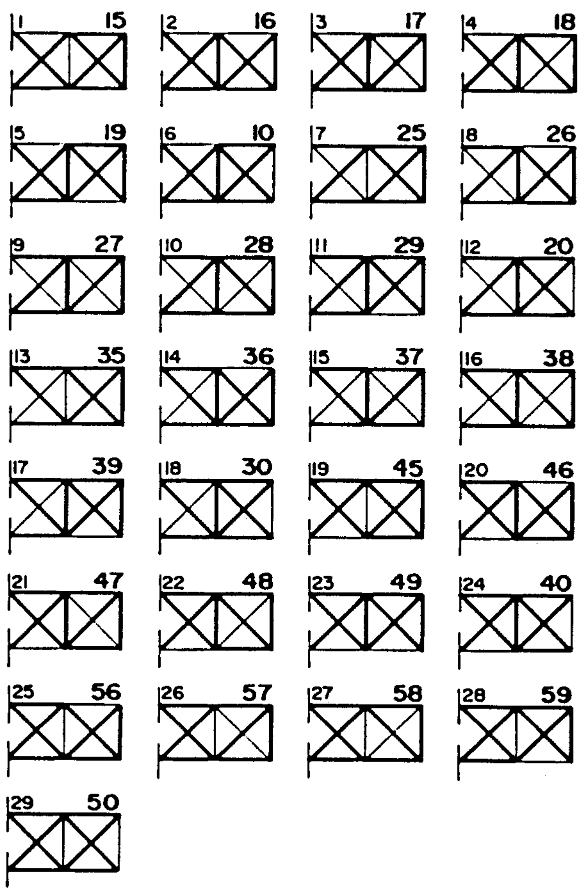

Fig. 3. The 29 stable statically determinate subsets of the ten-bar truss.

Table 2

Substructure		Internal loads in bars $[t_{ik}]$										B_k
Number $[k]$	Missing bars	1	2	3	4	5	6	7	8	9	10	
–	12	Unstable										
–	13	Unstable										
–	14	Unstable										
1	15	•	$3\sqrt{2}$	$\sqrt{2}$	-4	•	2	$-\sqrt{2}$	$-2\sqrt{2}$	1	2	1
2	16	•	$3\sqrt{2}$	$\sqrt{2}$	-4	-2	•	$\sqrt{2}$	0	-1	0	1
3	17	•	$3\sqrt{2}$	$\sqrt{2}$	-4	-1	1	•	$\sqrt{2}$	0	1	$2\sqrt{2}$
4	18	•	$3\sqrt{2}$	$\sqrt{2}$	-4	-2	0	$\sqrt{2}$	•	-1	0	$2\sqrt{2}$
5	19	•	$3\sqrt{2}$	$\sqrt{2}$	-4	-1	1	0	$-\sqrt{2}$	•	1	1
6	10	•	$3\sqrt{2}$	$\sqrt{2}$	-4	-2	0	$\sqrt{2}$	0	-1	•	1
	23	Unstable										
	24	Unstable										
7	25	3	•	$-2\sqrt{2}$	-1	•	-1	$2\sqrt{2}$	$\sqrt{2}$	-2	-1	$2\sqrt{2}$
8	26	3	•	$-2\sqrt{2}$	-1	1	•	$\sqrt{2}$	0	-1	0	$2\sqrt{2}$
9	27	3	•	$-2\sqrt{2}$	-1	2	1	•	$-\sqrt{2}$	0	1	8
10	28	3	•	$-2\sqrt{2}$	-1	1	0	$\sqrt{2}$	•	-1	0	8
11	29	3	•	$-2\sqrt{2}$	-1	2	1	0	$-\sqrt{2}$	•	1	$2\sqrt{2}$
12	20	3	•	$-2\sqrt{2}$	-1	1	0	$\sqrt{2}$	0	-1	•	$2\sqrt{2}$
	34	Unstable										
13	35	1	$2\sqrt{2}$	•	-3	•	1	0	$-\sqrt{2}$	0	1	$2\sqrt{2}$
14	36	1	$2\sqrt{2}$	•	-3	-1	•	$\sqrt{2}$	0	-1	0	$2\sqrt{2}$
15	37	1	$2\sqrt{2}$	•	-3	0	1	•	$-\sqrt{2}$	0	1	8
16	38	1	$2\sqrt{2}$	•	-3	-1	0	$\sqrt{2}$	•	-1	0	8
17	39	1	$2\sqrt{2}$	•	-3	0	1	0	$-\sqrt{2}$	•	1	$2\sqrt{2}$
18	30	1	$2\sqrt{2}$	•	-3	-1	0	$\sqrt{2}$	0	-1	•	$2\sqrt{2}$
19	45	4	$-\sqrt{2}$	$-3\sqrt{2}$	•	•	-2	$3\sqrt{2}$	$2\sqrt{2}$	-3	-2	1
20	46	4	$-\sqrt{2}$	$-3\sqrt{2}$	•	2	•	$\sqrt{2}$	0	-1	0	1
21	47	4	$-\sqrt{2}$	$-3\sqrt{2}$	•	3	1	•	$-\sqrt{2}$	0	1	$2\sqrt{2}$
22	48	4	$-\sqrt{2}$	$-3\sqrt{2}$	•	2	0	$\sqrt{2}$	•	-1	0	$2\sqrt{2}$
23	49	4	$-\sqrt{2}$	$-3\sqrt{2}$	•	3	1	0	$-\sqrt{2}$	•	1	1
24	40	4	$-\sqrt{2}$	$-3\sqrt{2}$	•	2	0	$\sqrt{2}$	0	-1	•	1
25	56	2	$\sqrt{2}$	$-\sqrt{2}$	-2	•	•	$\sqrt{2}$	0	-1	0	1
26	57	1	$2\sqrt{2}$	0	-3	•	1	•	$-\sqrt{2}$	0	1	$2\sqrt{2}$
27	58	2	$\sqrt{2}$	$-\sqrt{2}$	-2	•	0	$\sqrt{2}$	•	-1	0	$2\sqrt{2}$
28	59	1	$2\sqrt{2}$	0	-3	•	1	0	$-\sqrt{2}$	•	1	1
29	50	2	$\sqrt{2}$	$-\sqrt{2}$	-2	•	0	$\sqrt{2}$	0	-1	•	1
–	67	Unstable										
–	68	Unstable										
	69	Unstable										
–	60	Unstable										
–	78	Unstable										
–	79	Unstable										
–	70	Unstable										
–	89	Unstable										
–	80	Unstable										
–	90	Unstable										

We have chosen here to express the forces directly in terms of the cross-sectional areas of the elements, i.e. we are writing explicit expressions in terms of A_is rather than s_is. A typical π_{lm} represents the product of all the cross-sections of the bars except bars l and m. For example, π_{25} stands for the product $A_1A_3A_4A_6A_7A_8A_9A_0$ (note, A_2 and A_5 are missing). To determine the B_k coefficients one writes two compatibility equations, such as

$$e_1-\sqrt{2}e_2-\sqrt{2}e_3+e_4+e_5 = 0 \tag{33a}$$

$$e_5+e_6-\sqrt{2}e_7-\sqrt{2}e_8+e_9+e_{10} = 0. \tag{33b}$$

The element elongations are obtained by dividing each force by its stiffness. The numerators of the elongations in bars 1 and 2, for instance, are

$$\begin{aligned} e_1 &= 3B_{25}\pi_{125}+3B_{26}\pi_{126}+3B_{27}\pi_{127}+3B_{28}\pi_{128}+\cdots+2B_{50}\pi_{150} \\ e_2 &= 6B_{15}\pi_{125}+6B_{16}\pi_{126}+6B_{17}\pi_{127}+6B_{18}\pi_{128}+\cdots+2B_{50}\pi_{150}, \end{aligned} \tag{34}$$

where π_{lmn} represents the product of all the cross-sections of the structure except bars l, m and n. Introducing the bar elongations into eqns (33) and grouping by the π_{lmn} products, yields two equations from which the B_k coefficients will be determined. The first compatibility equation (33a) for instance becomes

$$(3B_{25}-6\sqrt{2}B_{15})\pi_{125}+(3B_{26}-6\sqrt{2}B_{16})\pi_{126}+\cdots = 0. \tag{35}$$

Since this relation must be satisfied for any values of the π_{lmn} products we obtain conditions of the form

$$\begin{aligned} B_{25}-2\sqrt{2}B_{15} &= 0 \\ B_{26}-2\sqrt{2}B_{16} &= 0. \end{aligned} \tag{36}$$

This leads to an overdetermined but consistent set of homogeneous linear equations in the 29 B_k coefficients, out of which only 28 equations are independent equations. For instance, setting $B_{15} = 1$ one obtains the other 28 coefficients by simple back-substitution. The result is given in the last column of Table 2.

6. CONCLUSIONS

This paper has presented the analytic expression of the internal forces in a linear elastic truss, subjected to static loads, as an explicit function of the stiffnesses of the bars. The three basic ingredients of structural analysis, that is, equilibrium, constitutive law and compatibility of deformations appear in the equations in a highly structured manner. Instead of assembling the stiffness matrix, as is done in the matrix displacement method, and solving the equilibrium equations, we now have a technique to assemble directly the solution. The explicit analysis equations were developed for trusses. However, Fuchs (1991) has shown that the analysis of any framed structure, including bending elements, can be cast in the form of a truss [eqns (1)] by the uncoupling of the bending deformation into a pure moment mode and a "pure" shear mode. This leads to a diagonal stiffness matrix S which is what characterizes the truss. Consequently, the present method can be extended to include structures composed of flexural elements.

This brings us to the issue of implementation. The proposed method is based on scanning the statics matrix Q in order to determine the set of statically determinate stable substructures and solving for the internal loads in every substructure. For this purpose we have at our disposal well established numerical techniques which are used in the revised simplex method of linear programming (Strang, 1986). To determine the compatibility constants B_k one needs to write the set of relations in the B_ks, which leads to the solution of a system of linear equations whose rank is now known. The algorithms for applying the method should pose no problem. The only questionable aspect is the number of terms involved in the explicit expression. In many cases they will render the explicit solution intractable.

The main contribution of this paper is the presentation of the analytic solution of the structural analysis equations. For small size structures it constitutes an efficient solution for structural reanalysis. For large engineering structures one must still consider approximate analysis models. However, a perusal of the analytic expressions, may help in deriving improved approximations which will be based on structural considerations, in contrast to the prevailing approach, which is mathematical in essence.

REFERENCES

Abu Kassim, A. M. and Topping, B. H. V. (1985). Static Reanalysis of Structures: A Review, in Civil-Comp, *Proc. Sec. Int. Conf. Civ. and Str. Eng. Comp.*, London (Edited by B. H. V. Topping), 2. Civil-Comp Press, Edinburgh GB, 137–146.
Arora, J. S. (1976). Survey of structural reanalysis techniques. *J. Str. Div. ASCE* **102**, 783–802.
Asplund, S. O. (1966). *Structural Mechanics: Classical and Matrix Methods*. Prentice-Hall, Englewood Cliffs, NJ.
Ayres, F. (1962). *Theory and Problems of Matrices, Schaum's Outline Series*. McGraw-Hill, New York.
Fuchs, M. B. (1981). Explicit optimum design. *Int. J. Solids Structures* **18**(1), 13–22.
Fuchs, M. B. (1991). Unimodal beam elements. *Int. J. Solids Structures* **27**(5), 533–545.
Kirsch, U. (1981). Optimum Structural Design. McGraw-Hill, New York.
Strang, G. (1986). *Introduction to Applied Mathematics*. Wellesley-Cambridge, Massachusetts.

APPENDIX: THE B_{lk} COEFFICIENTS DEPENDENT ONLY ON THE GEOMETRY OF THE STRUCTURE

In Section 4 of this paper, the method for obtaining the linear equations in the B_{lk} compatibility coefficients was performed analytically for the three-bar truss problem. In the case of the three-bar truss the external loading (p_1, p_2) vanished from the compatibility equations (30). We will show here that this is a general result.

Consider a compatibility condition resulting from the product π_{lmn} in the ten-bar truss (see eqn 35). The compatibility coefficients appearing in that condition originate from the terms in B_{lm}, B_{mn} and B_{ln} in the expression of the bar elongations e_n, e_l and e_m respectively. Consequently the compatibility condition takes the form

$$a_{lm}B_{lm}t_{n,lm} + a_{mn}B_{mn}t_{l,mn} + a_{ln}B_{ln}t_{m,ln} = 0, \tag{A1}$$

where the a_{lm} are constants depending on geometry only, and $t_{n,lm}$ is typically the force in bar n for the subset lm. The forces in the above equation are related to the external loading through

$$\begin{aligned} t_{n,lm} &= h^{\mathrm{T}}_{n,lm}p \\ t_{l,mn} &= h^{\mathrm{T}}_{l,mn}p \\ t_{m,ln} &= h^{\mathrm{T}}_{m,ln}p, \end{aligned} \tag{A2}$$

where $h_{l,mn}$ is typically the vector of the elements of row l in the inverse of the statics matrix Q_{mn} of subset (mn). Note, the matrices Q_{lm} and Q_{ln} are obtained from matrix Q_{mn} by removing column l and replacing it by columns n and m respectively (the column indices refer to their position in the statics matrix Q). Using the product form of the inverse (Strang, 1985) one can show that

$$\begin{aligned} h_{n,lm} &= h_{l,mn}/v_{ln} \\ h_{m,ln} &= h_{l,mn}/v_{lm}, \end{aligned} \tag{A3}$$

where the scalars v_{ln} and v_{lm} are the lth entry of the vector v in the system of equations (A4a) and (A4b) respectively:

$$Q_{mn}v = q_n \tag{A4a}$$

$$Q_{mn}v = q_m. \tag{A4b}$$

Since the three h vectors in eqn (A2) are parallel vectors their scalar products with p will cancel out from the homogeneous equations (A1). Consequently, the B_{lk} coefficients are independent from the external loading. They reflect only the conditions for compatible deformations for any combination of stiffnesses of the bars of the structure.